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Vorticity and non-coaxiality in progressive deformations

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Abstract—A measure of the non-coaxiality involved in progressive deformation histories is proposed in the form of the kinematical vorticity number, W_k. This number is a measure of the relative effects of rotation of material lines (relative to the instantaneous stretching axes) and of stretching of these material lines. As such, W_k, is a measure of the instantaneous degree of non-coaxiality. A detailed example is first presented in the form of a progressive simple shearing in which the shear plane rotates relative to an external coodinate system. This is followed by examples of more complicated deformation histories. Three specific types of progressive, isochoric (constant volume) deformation histories are recognized. Those for which $0 \leq W_k < 1$ correspond to deformation histories where no line that has been extended is shortened in future increments; $W_k = 0$ is a special case of these corresponding to a coaxial history. Histories with $W_k > 0$ are non-coaxial. Those histories with $W_k = 1$ correspond to progressive simple shearing. Those histories with $1 < W_k < \infty$ are pulsating and lines that have been extended may be shortened in future increments.

INTRODUCTION

IN NON-COAXIAL progressive deformations such as occur on fold limbs or in shear zones, material lines rotate through the principal directions of the rate-of-deformation tensor. This rotation can be reflected in rock fabrics and students of fabric therefore make an important distinction between coaxial and non-coaxial progressive deformations (e.g. Elliott 1972, Hobbs *et al.* 1976, Williams 1976, Lister 1976). Further development of this idea requires that various degrees of non-coaxiality be distinguished perhaps along the lines suggested by Elliott (1972 p. 2628). The purposes of this paper are to discuss quantities used to describe the rotational properties of progressive deformations and to point to the kinematical vorticity number of Truesdell (1953) as an additional measure of non-coaxiality that may be useful in fabric studies. The discussion centres around a single, specific, geologically realistic example of a non-coaxial progressive deformation, namely a progressive simple shear with rotation of the shear planes in an external frame of reference. This example is discussed first in terms of angular velocities and then in terms of vorticity vectors and vorticity and stretching tensors. General deformation histories are discussed at the end of the paper. We have profited from reading a manuscript by McKenzie (1979) prior to publication in which he independently employs a measure of non-coaxiality that is equivalent to the kinematical vorticity number of Truesdell.

AN EXAMPLE

Figure 1(a, b) represents a situation in which a region *abcd* in a rock body is undergoing simple shearing with respect to the coordinate axes **x** while these axes in turn are rotating in an external coordinate system **X**. In nature the **X** directions might be geographic directions and the x_1x_3 plane might be parallel to the boundaries of a shear zone, or the X_2X_3 plane might correspond to the axial plane of a fold and the x_1x_3 plane might be bedding. The **x** axes are taken to be rotating clockwise in the **X** frame at a constant rate (of 1 rad per unit time) while the simple shearing is proceeding counterclockwise, at the

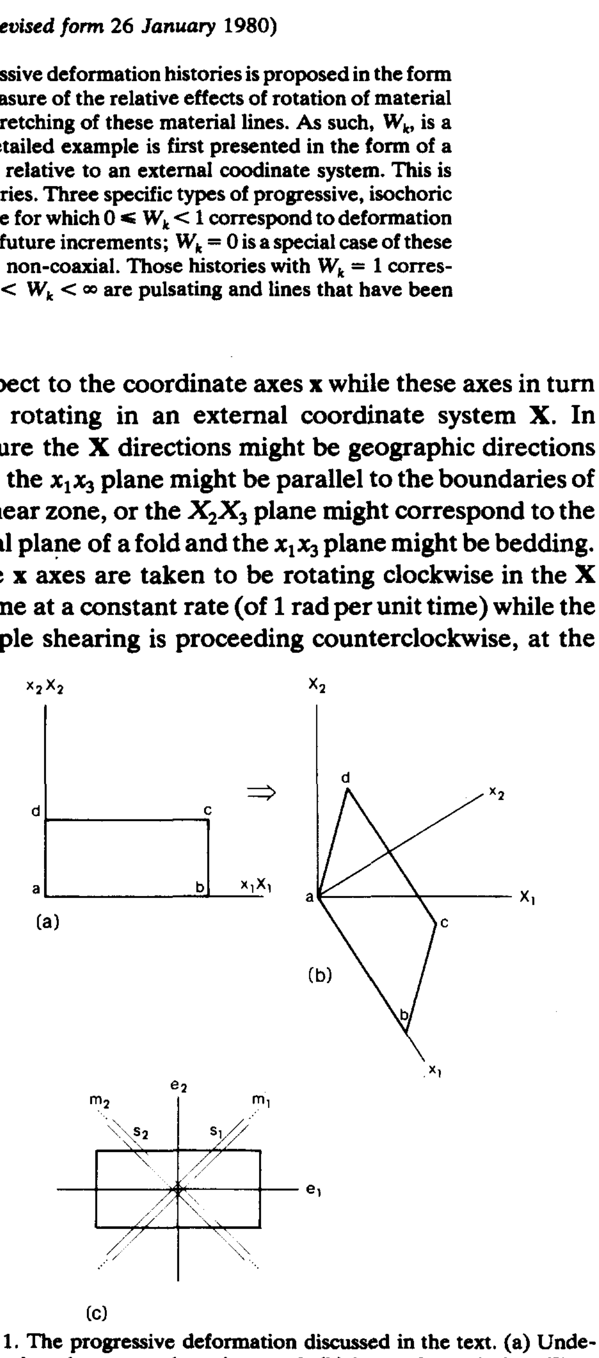

Fig. 1. The progressive deformation discussed in the text. (a) Undeformed marker rectangle at time $t = 0$. (b) Same after unit time. The **x** axes are fixed parallel and perpendicular to *ab* and rotate with this material line. The positive ends of the X_3 and x_3 axes extend towards the viewer, perpendicular to the page. (c) Marker rectangle in its initial position, with the stretching axes (**s**), material lines under them (**m**) and external axes (**e**) indicated.

same rate, and such that the engineering shear strain-rate of *ab* is 1 per unit time.

At any given moment during the progressive deformation, for example, in the first moment (Fig. 1c), we distinguish three sets of orthogonal lines in the plane of the paper. One set comprises the principal directions of the rate-of-deformation tensor (also the principal directions of incremental strain), hereafter called the instantaneous stretching axes (**s**). These lie at ± 45° to the x_1 direction throughout the progressive deformation. The second set of lines comprises material lines (**m**) instantaneously "under" (parallel to) the stretching directions. The important property of the **m** lines for our purpose is that they are the only pair of orthogonal lines in the plane of the paper that have shear strain rate of zero with respect to one another. The **m** lines thus act as if they are rigidly attached to each other, in the sense that they rotate at the same rate and in the same direction as each other. The third set of orthogonal lines comprises the external coordinate directions (**e**), parallel in this case to the **X** axes.

Angular velocities

The most elementary way to describe rotational features of the progressive deformation is to use the angular velocities relating the sets of lines **s**, **m** and **e** to one another. An angular velocity is represented by an axial vector with direction parallel to the axis of rotation and magnitude equal to the angular speed. We take the sense of the vector and the sign of its magnitude to be indicated by a right-hand rule. The angular velocity of the **x** axes with respect to the **X** axes is thus −1 rad per unit time, and it is represented by a vector of unit length pointing in the direction of the negative X_3 axis (into the plane of Fig. 1). This is also the angular velocity of the **s** lines relative to the **X** = **e** directions, since the **s** lines are fixed relative to the **x** directions. We can write

$$\boldsymbol{\omega}_{em} + \boldsymbol{\omega}_{ms} + \boldsymbol{\omega}_{se} = \mathbf{O}, \tag{1}$$

where $\boldsymbol{\omega}_{em}$ represents the instantaneous angular velocity of the **m** lines with respect to the **e** lines, etc. Transposing terms and suffixes, this becomes

$$\boldsymbol{\omega}_{em} = \boldsymbol{\omega}_{es} + \boldsymbol{\omega}_{sm}. \tag{2}$$

The angular velocity $\boldsymbol{\omega}_{em}$ can thus be thought of as consisting of two components, the angular velocity of the instantaneous stretching axes in the external frame and the angular velocity of the **m** lines relative to the instantaneous stretching axes. The importance of this for fabric studies is that the first component has no influence on fabric development (if we overlook fabric features that are sensitive to the orientation of a deforming rock in the gravity or magnetic fields). The important angular velocity for fabric development is the $\boldsymbol{\omega}_{sm}$ component. When this is zero a progressive deformation is ideally coaxial, and when it is non-zero the progressive deformation is non-coaxial. In our example, ω_{sm} is 0.5, ω_{es} is −1, and ω_{em} is −0.5, all in radians per unit time. The **m** lines are rotating clockwise in the **e** frame at the same rate that they are rotating counterclockwise in the **s** frame.

Vorticity vectors

The vorticity vectors here are axial vector quantities closely related to the angular velocities $\boldsymbol{\omega}_{em}$ and $\boldsymbol{\omega}_{sm}$ just described. In short, there is a vorticity, which we shall call **W**, of the material with respect to the external axes and a different vorticity **w** of the material with respect to the stretching axes, with

$$\begin{aligned} \mathbf{W} &= 2\boldsymbol{\omega}_{em} \\ \mathbf{w} &= 2\boldsymbol{\omega}_{sm} \end{aligned} \tag{3}$$

As these equations indicate, the vorticity vectors are parallel to the respective angular velocity vectors and have double their magnitudes. For our example $W = -1$ and $w = 1$, in radians per unit time.

Equations (3), which are applicable to this particular example, arise from the general definition of the vorticity, which is that the vorticity is the curl of a velocity field or, using our notation,

$$\begin{aligned} \mathbf{W} &\equiv \text{curl } \mathbf{V} \\ \mathbf{w} &\equiv \text{curl } \mathbf{v} \end{aligned} \tag{4}$$

where **V** represents velocities as seen from the external axes and **v** represents the different velocities seen from the coordinate system **x**. In our example this is equivalent to the velocity as seen from the frame of the stretching axes. We return to this fundamental definition later when we calculate **W** and **w** from the respective velocity fields. As for the angular velocities, the vorticity with respect to external axes (**W**) has no simple relationship to fabric evolution, but the vorticity with respect to the stretching axes (**w**) is important.

Several interpretations of the vorticity are available (see Truesdell 1954, pp. 59–65). With respect to Fig. 1, one of these says that the vorticity vector perpendicular to the page has magnitude equal to two times the instantaneous mean rate of right-handed (counter-clockwise) rotation of all material lines lying in the plane of the paper. Another interpretation, as applied to our example, is that the vorticity vector perpendicular to the page has magnitude equal to the sum of the rates of right-handed rotation of any pair of orthogonal material lines in the plane of the paper. The **m** lines are such a pair, so the vorticities in the two reference frames are given by

$$\begin{aligned} \mathbf{W} &= \boldsymbol{\omega}_{em_1} + \boldsymbol{\omega}_{em_2} \\ \mathbf{w} &= \boldsymbol{\omega}_{sm_1} + \boldsymbol{\omega}_{sm_2} \end{aligned}$$

which are equivalent to equations (3) since the two **m** lines (m_1 and m_2) have identical angular velocities.

By doubling both sides of (2) and substituting (3) we obtain

$$\mathbf{W} = 2\boldsymbol{\omega}_{es} + \mathbf{w}$$

or

$$\mathbf{W} = \mathbf{W}' + \mathbf{w} \tag{5}$$

if we set $\mathbf{W}' \equiv 2\boldsymbol{\omega}_{es}$. We have given $\mathbf{W}'$ the same kind of symbol as a vorticity, but it is really not a vorticity, only two times an angular velocity. Vorticities, strictly speaking, exist only for velocity fields that extend in two or three dimensions, whereas angular velocities require only one-dimensional arrays of velocities, as in the case of the lines defining our **s** directions. A suitable name for **W**′, is the spin. This term is used in a variety of ways by Truesdell & Toupin (1960, pp. 353–357), but in each case it refers to an angular velocity of some kind of line. On their p. 355, spin is used specifically for the angular velocity of the instantaneous stretching axes. While spin seems a suitable short name for **W**′, a more complete and correct explanation would be that **W**′ is twice the spin of the instantaneous stretching axes in the external frame. In view of this, we call **W**′ the spin component. The quantities **w** and **W** in (5) are both vorticities, and it has been suggested by W. S. F. Kidd (pers. comm.) that they be distinguished as the internal and external vorticities respectively.

Using these terms, (5) could be stated in words as: the external vorticity is the sum of a spin component and the internal vorticity. The internal vorticity is the important kind of vorticity for fabric development.

The above definitions of **W**′ and **w** provide a basis for the four-fold classification of progressive deformations shown in Fig. 2. When the internal vorticity **w** is zero, a progressive deformation is instantaneously coaxial. If this condition persists over a finite period of time, we have what is usually meant by a coaxial progressive deformation. When the spin component **W**′ is zero a progressive deformation is one in which the stretching axes are instantaneously not rotating in the external reference frame. It is tempting to call this situation 'irrotational', but this would run counter to the usual practice in continuum mechanics (e.g. Mase 1970, p. 113, Malvern 1969, p. 148), where an irrotational flow is any flow for which the vorticity vanishes. The nature of the vorticity is not specified. It may be the internal vorticity **w**, in which case irrotational becomes a permissible synonym for coaxial, or an external vorticity, in which case irrotational is not a suitable synonym for coaxial. For lack of better terms we have used spinning and non-spinning in Fig. 2, to refer to the classes where $\mathbf{W}' \neq 0$ and $\mathbf{W}' = 0$ respectively. It has to be kept in mind that the spin referred to here is specifically the spin of the instantaneous stretching axes in the external reference frame. Whatever the terms used for the classes, **W**′ and **w** are distinct, well-defined and significant quantities on which a four-fold classification can be used.

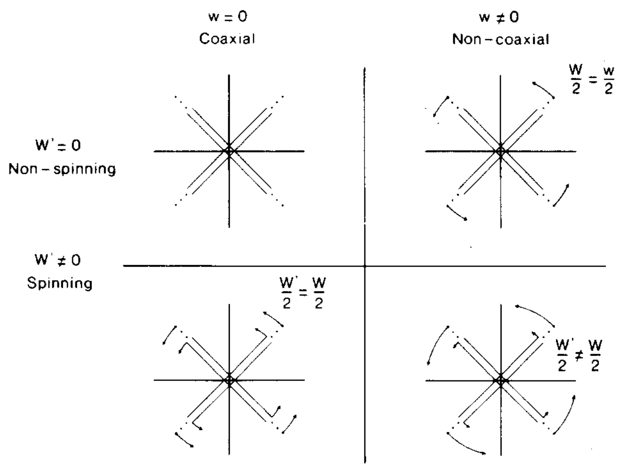

Fig. 2. Four classes of progressive deformation, defined by the magnitudes of **w** (the internal vorticity) and **W**′ (the spin component of the external vorticity). Arcs with arrow heads indicate possible angular speeds of the **s** (double) lines and the **m** (dotted lines) in the frame of the **e** (single) lines.

Calculation of vorticities from the velocity fields

The spatial description of the velocity field with respect to the **x** axes in our example is given by

$$\begin{aligned} v_1 &= 0x_1 - 1x_2 + 0x_3 \\ v_2 &= 0x_1 + 0x_2 + 0x_3 \\ v_3 &= 0x_1 + 0x_2 + 0x_3. \end{aligned} \tag{6}$$

The nine coefficients of the x_j on the right hand side of the equations, the $\partial v_i/\partial x_j$, are the velocity gradients, and they make up the components of the velocity gradient tensor, to which we return later. The vorticity $\mathbf{w} \equiv \text{curl}\, \mathbf{v}$ is found by taking the vector cross product $\nabla \times \mathbf{v}$, which is given by the determinant

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ v_1 & v_2 & v_3 \end{vmatrix} \tag{7}$$

where **i**, **j** and **k** are unit vectors pointing in the directions of the positive x_1, x_2 and x_3 axes. Fortunately, in most cases to be discussed here, involving plane velocity fields with v_3 (or V_3) and $\partial/\partial x_3$ (or $\partial/\partial X_3$) equal to zero, determinant (7) reduces to

$$\mathbf{k}\left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right). \tag{8}$$

$\partial v_2/\partial x_1$ and $-\partial v_1/\partial x_2$ are respectively the rates of right-handed rotation of the material lines instantaneously under the x_1 and x_2 axes. Since these material lines are instantaneously perpendicular, we can see that the vorticity magnitude given by (8),

$$w = \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right) \tag{9}$$

corresponds to the second interpretation of the vorticity explained above. In our example, $\partial v_2/\partial x_1 = 0$ and $\partial v_1/\partial x_2 = -1$, so $w = 1$, as previously stated.

To obtain **W** by evaluating the curl of its velocity field we first need equations like (6) that describe the velocities relative to the **X** axes. The coefficients for this set of equations are found by adding the coefficients for the rigid rotation of the **x** axes in the **X** frame to the coefficients for the simple shearing part of the total deformation, after converting the latter to coefficients in the **X** coordinate system. The conversion is done using the transformation formula for second-order tensors, since the coefficients or velocity gradients in (6)

$$\frac{\partial v_i}{\partial x_j} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{10}$$

describe such a tensor. The transformation coefficients are the direction cosines l_{ij} relating the **X** axes to the **x** axes, and at any time t these are

$$l_{ij} = \begin{bmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{11}$$

Employing these cosines in the transformation formula, the velocity gradients relative to the **X** axes for the simple shearing part of the total deformation are found to be

$$\begin{bmatrix} -\cos t \ \sin t & -\cos^2 t & 0 \\ \sin^2 t & \sin t \ \cos t & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{12}$$

When these gradients are added to the gradients for the rigid rotation of the **x** axes relative to the **X** axes,

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{13}$$

the total velocity gradients are obtained and these are the coefficients in the equations describing the velocity field at any time t with respect to the **X** axes:

$$\left.\begin{aligned} V_1 &= (-\cos t \sin t)\, X_1 + (1-\cos^2 t)\, X_2 + 0X_3 \\ V_2 &= (-1 + \sin^2 t)\, X_1 + (\sin t \cos t)\, X_2 + 0X_3 \\ V_3 &= 0X_1 + 0X_2 + 0X_3 \end{aligned}\right\} \tag{14}$$

The vorticity $\mathbf{W} = \text{curl}\, \mathbf{V}$ can then be evaluated as before and

$$W = -1 \tag{15}$$

as we determined earlier.

Vorticity and stretching tensors

A third way to discuss our example involves further use of second-order tensor quantities. This is valuable because it shows how vorticity and the rate of straining are related and leads on to a measure of non-coaxiality.

The velocity gradients in (6) or (14) are the components of two velocity gradient tensors and each of these can be decomposed as follows into a symmetric and an anti-symmetric part.

$$\underset{\text{Velocity gradients tensor}}{\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}} = \underset{\text{Stretching tensor}}{\begin{bmatrix} 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}} + \underset{\text{Vorticity } (\mathbf{w}) \text{ tensor}}{\begin{bmatrix} 0 & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}} \tag{16}$$

The symmetric part is called the rate of deformation tensor or the stretching tensor and the anti-symmetric part is the vorticity tensor or the spin tensor (e.g. Malvern 1969, p. 145; Mase 1970, p. 112). Note that despite its name the stretching tensor has diagonal components which express longitudinal strain-rates in terms of elongations, not stretches, per unit time.

Because it is defined as the anti-symmetric part of the velocity gradient tensor, the vorticity tensor **W** or **w** can never have more than three independent components, e.g. $W_{12} = -W_{21}$, $W_{23} = -W_{32}$ and $W_{31} = -W_{13}$, and these are always simply related to the components of the vorticity vector and to the spins (angular velocities) of material lines under the axial directions as follows,

$$\underset{\text{Components of vorticity tensor}}{\begin{bmatrix} 0 & w_{12} & w_{13} \\ w_{21} & 0 & w_{23} \\ w_{31} & w_{32} & 0 \end{bmatrix}} = \underset{\text{Same, given in terms of components of vorticity vector } (w_1\ w_2\ w_3)}{\begin{bmatrix} 0 & -\frac{1}{2}w_3 & \frac{1}{2}w_2 \\ \frac{1}{2}w_3 & 0 & -\frac{1}{2}w_1 \\ -\frac{1}{2}w_2 & \frac{1}{2}w_1 & 0 \end{bmatrix}} =$$

$$\underset{\text{Same, given in terms of angular velocities}}{\begin{bmatrix} 0 & -\frac{1}{2}(\omega_{21} + \omega_{12}) & \frac{1}{2}(\omega_{13} + \omega_{31}) \\ \frac{1}{2}(\omega_{12} + \omega_{21}) & 0 & -\frac{1}{2}(\omega_{23} + \omega_{32}) \\ -\frac{1}{2}(\omega_{31} + \omega_{13}) & \frac{1}{2}(\omega_{32} + \omega_{23}) & 0 \end{bmatrix}} \tag{18}$$

where ω_{ij} means the right-handed angular velocity with respect to the **i** axis of the material line instantaneously under the **j** axis.

The vorticity tensor, the vorticity vector and the sums of the spins $(\omega_{ij} + \omega_{ji})$ thus describe precisely the same rotational properties of the field. One can see from the first and third parts of (18) why the vorticity tensor is also reasonably called the spin tensor; its components have absolute magnitudes which are average spins of pairs of material lines under the coordinate axes.

MEASURES OF NON-COAXIALITY

Since $\omega_{sm} = \mathbf{O}$, or $\mathbf{w} = \mathbf{O}$ are two ways of defining instantaneous perfect coaxiality, the magnitudes of either of these quantities would seem to make reasonable measures of the degree of non-coaxiality. This kind of measure is one of the possible measures of non-

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$$\begin{matrix} -\cos t \sin t & \\ & \sin t \cos t \end{matrix} = \underset{\text{Stretching tensor}}{\begin{bmatrix} -\cos t \ \sin t & \dfrac{-\cos 2t}{2} & 0 \\ \dfrac{-\cos 2t}{2} & \sin t \ \cos t & 0 \\ 0 & 0 & 0 \end{bmatrix}} + \underset{\text{Vorticity } (\mathbf{W}) \text{ tensor}}{\begin{bmatrix} 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}} \tag{17}$$

coaxiality suggested by Elliott (1972, p. 2628). We can develop this suggestion further by making explicit that it is not the magnitude of the internal vorticity alone that can be reflected in the fabric, but the relative magnitudes at any instant of the internal vorticity and the stretching (the rate of straining). Thus, the important factor in the development of the fabric is the instantaneous rate at which material lines parallel to the instantaneous stretching axes are rotating with respect to those axes relative to the rate at which these material lines are being stretched along these axes. In a coaxial deformation history the same material lines remain parallel to the instantaneous stretching axes from one instant to the next. In a non-coaxial deformation history different lines of material particles coincide with the instantaneous stretching axes from one instant to the next. Here it is not solely the rate of rotation of lines of material particles relative to the instantaneous stretching axes that is important; this rate of rotation relative to the rate of stretching governs fabric evolution. A convenient way to describe these relative magnitudes is to use the kinematical vorticity number W_k of Truesdell (1953), defined for our purposes (where **w** not **W** is the important vorticity) as

$$W_k \equiv \frac{w}{(2\,\overline{II})^{1/2}} = \frac{w}{\left[2(s_1^2 + s_2^2 + s_3^2)\right]^{1/2}} \qquad (19)$$

Here w is the magnitude of the vorticity vector, the s_i are the principal stretchings (strain-rates), and $\overline{II}$ is the second moment of the stretching tensor (Ericksen 1960, pp. 833–834), an invariant quantity related to the ordinary first and second invariants by

$$\overline{II} = I^2 + 2II \qquad (20)$$

$\overline{II}^{1/2}$ can be taken as a measure of the intensity of any second-order tensor, and it is used in (19) as a measure of the intensity of the stretching. ($\overline{II}$, can also be written as $\mathrm{tr}(ss^T)$, the trace of the tensor formed by multiplying the stretching tensor by its transpose.) The initial 2 in (19) is inserted so that the kinematical vorticity number for simple shearing has an absolute magnitude of exactly 1.

The stretching tensor in our example has components

$$\begin{bmatrix} 0 & -\tfrac{1}{2} & 0 \\ -\tfrac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

relative to the **x** axes, or components

$$\begin{bmatrix} -\tfrac{1}{2} & 0 & 0 \\ 0 & \tfrac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

relative to the stretching axes. The vorticity vector **w** has magnitude 1, so

$$W_k = \frac{1}{\left[2(\tfrac{1}{4} + \tfrac{1}{4} + 0)\right]^{1/2}} = 1.$$

The absolute value of the kinematical vorticity number can range from zero towards infinity. It is zero for perfect coaxiality, 1 for simple shearing where the vortical and stretching components of the velocity gradient tensor are equal, and > 1 where the vortical component is larger than the stretching component. For behaviour approaching that of rigid-body rotation, where the vortical component is very large compared with the stretching component, W_k approaches infinity. For ideal rigid body behaviour all sets of orthogonal material lines are principal directions of the stretching tensor and there is no rotation with respect to any of them or of the material lines under them. So the vorticity w and the stretching intensity both became zero and the kinematical vorticity number is indeterminate.

It is to be noted that the kinematical vorticity number is a measure of the degree of non-coaxiality at an instant in the deformation history.

McKenzie (1979) independently uses the kinematical vorticity number in a two-dimensional investigation of how the finite strain accumulates under various combinations of stretching and vorticity. He picks reference axes like our **x** axes, at 45° to the stretching directions, and can then write the velocity gradients for isochoric (constant volume) progressive deformation as

$$\begin{bmatrix} 0 & (s - w) \\ (s + w) & 0 \end{bmatrix}$$

where the s's represent the stretching component and the w's the vortical component of the velocity gradients. He then simply takes the ratio $|w|/|s|$ as a measure of the relative intensities of these two components. This is exactly the same number as the kinematical vorticity number since under these conditions w is always half the numerator of (19) and $|s|$ is always half the denominator. The ratio of $|w|$ to $|s|$ is therefore a suitable definition of the kinematical vorticity number for isochoric progressive deformation in two dimensions, with axes chosen as above, whereas (19) provides the general definition, suitable for three-dimensional situations and any choice of axes.

MORE COMPLICATED PLANE STRAIN DEFORMATION HISTORIES

The previous discussion has considered a deformation history consisting of a progressive simple shear for which $W_k = 1$ throughout the history. The purpose of this section is to consider more complicated histories for which W_k may be greater than or smaller than 1.

A general deformation history involving plane strain is represented by equations of the form

$$x_1 = f_1\,(X_1, X_2, t)$$
$$x_2 = f_2\,(X_1, X_2, t)$$

where the X_i, and x_i are the coordinates of a particle in the undeformed and deformed states respectively, t is time and f_1, f_2 are linear or non-linear functions. The general form of these equations does not seem to have been investigated, although Ramberg (1975) has considered progressive deformation which may be thought of as the simultaneous operation of pure shear increments together with simple shear increments in which the simple shear plane is inclined at an arbitrary angle, θ, to a principal axis of the pure shear. Ramberg concentrates on situations where the rate of pure shear, $\dot{\varepsilon}$, the rate of simple shear $\dot{\gamma}$ and θ are constant throughout the history. In solving the differential equations for the rate of displacement during these deformations, Ramberg distinguishes three types of deformation history:

Type 1 deformation histories (non pulsating; $0 \leq W_k < 1$)

The deformation history may be described by

$$x_1 = [A\beta_{11} + \exp(-\alpha t)] X_1 + \beta_{12} A X_2$$
$$x_2 = -\beta_{21} A X_1 + [\exp(\alpha t) - A\beta_{11}] X_2$$

where $A = \exp(\alpha t) - \exp(-\alpha t)$ and, for isochoric deformation histories, $\beta_{12}\beta_{21} = \beta_{11}(\beta_{11} - 1)$. If one chooses to think of this type of deformation history as resulting from the simultaneous operation of simple and pure shears then the constants α, β_{11}, β_{12}, β_{21} may be written in terms of $\dot{\varepsilon}$, $\dot{\gamma}$, and θ [see Ramberg 1975, equations (31), (38) and (39)]. In particular,

$$\alpha = +\tfrac{1}{2}(\dot{\varepsilon}^2 - 2\,\dot{\varepsilon}\,\dot{\gamma}\sin\theta\cos\theta)^{\frac{1}{2}}$$

where, for type 1 histories $\dot{\varepsilon}^2 > 2\dot{\varepsilon}\,\dot{\gamma}\sin\theta\cos\theta$.

The velocity field for isochoric histories may be written in terms of the coordinates in the deformed state as

$$v_1 = \alpha(2\beta_{11} - 1)x_1 + 2\beta_{12}\alpha x_2$$
$$v_2 = -2\beta_{21}\alpha x_1 + \alpha(1 - 2\beta_{11})x_2$$

from which the kinematical vorticity number is

$$W_k = \frac{\beta_{12} + \beta_{21}}{2\,[(2\beta_{11} - 1)^2 + (\beta_{12} + \beta_{21})^2]^{\frac{1}{2}}}.$$

Thus, these histories are coaxial whenever

$$\beta_{12} = \beta_{21} = 0 \text{ or } \beta_{12} = -\beta_{21}.$$

It may be verified that $0 \leq W_k < 1$. These types of deformation histories correspond to the ones commonly considered in structural geology. The particle path lines are curved in general and although there exist lines of material particles that have undergone shortening in the past but will be extended in the future, there are no material lines that, having been extended in the past, will be shortened in future increments. An example is given by Ramberg (1975, fig. 3).

Type 11 deformation histories (pulsating; $1 < W_k < \infty$)

The deformation history may be described by

$$x_1 = [\cos\beta t + \frac{a_{11}}{\beta}\sin\beta t]\, X_1 + \frac{a_{12}}{\beta}\sin\beta t\, X_2$$

$$x_2 = \frac{a_{21}}{\beta}\sin\beta t\, X_1 + [\cos\beta t + \frac{a_{22}}{\beta}\sin\beta t]\, X_2$$

where, for isochoric deformations, $a_{11} = -a_{22}$, $a_{11}a_{22} = a_{12}a_{21} + \beta^2$. Again, if one wishes, the constants may be expressed in terms of the simultaneous operation of simple and pure shear components (Ramberg 1975, pp. 43–44) and these histories correspond to the situation where

$$\dot{\varepsilon}^2 < 2\,\dot{\varepsilon}\,\dot{\gamma}\sin\theta\cos\theta.$$

The velocity field for isochoric deformations may be written in terms of the coordinates in the deformed state as

$$v_1 = a_{11}x_1 + a_{12}x_2$$
$$v_2 = a_{21}x_1 - a_{11}x_2$$

from which the kinematical vorticity number is

$$W_k = \frac{(a_{12} - a_{21})}{[(a_{12} - a_{21})^2 - 4\beta^2]^{\frac{1}{2}}}.$$

Notice that W_k is always greater than unity; for $\beta = 0$ (when $W_k = 1$) the deformation is not defined. These deformation histories are always non-coaxial and correspond to the 'pulsating' histories discussed by Ramberg (1975) and by McKenzie (1979). For these deformation histories material lines that are being extended may be shortened in future increments of deformation. This is not possible in type 1 or type 111 histories.

Type 111 deformation histories (simple shearing; $W_k = 1$).

$$x_1 = (1 + \beta_{11}t)X_1 + \beta_{12}t X_2$$
$$x_2 = \beta_{21}t X_1 + (1 + \beta_{22}t)X_2$$

where, for isochoric histories, $\beta_{11} = -\beta_{22}$, $\beta_{12}\beta_{21} = -\beta_{11}^2$ and again, these constants may be expressed in terms of simultaneous simple and pure shear components if one wishes (Ramberg 1975, pp. 47–48). These deformation histories correspond to the situation where

$$\dot{\varepsilon}^2 = 2\,\dot{\varepsilon}\,\dot{\gamma}\sin\theta\cos\theta.$$

The velocity field for isochoric histories may be written in terms of the coordinates in the deformed state as

$$v_1 = \beta_{11}x_1 + \beta_{12}x_2$$
$$v_2 = \beta_{21}x_1 - \beta_{11}x_2$$

from which it may be confirmed that W_k is always equal to unity. This type of deformation history corresponds to progressive simple shear, relative to some set of axes; particle paths are always straight relative to these axes (see Ramberg 1975, fig. 10 for an example).

It can be seen, then, that progressive simple shearing ($W_k = 1$) is a history that is intermediate between the pulsating histories ($1 < W_k < \infty$) and the non-pulsating histories ($0 \leq W_k < 1$); $W_k = 0$ corresponds to a special type of history which is coaxial. These relationships are illustrated in Fig. 3.

The discussion may be extended to three dimensions and Ramberg (1975) has shown that there exist deformations for which the history is pulsating with respect to one pair of principal axes of strain and non-pulsating with respect to another pair.

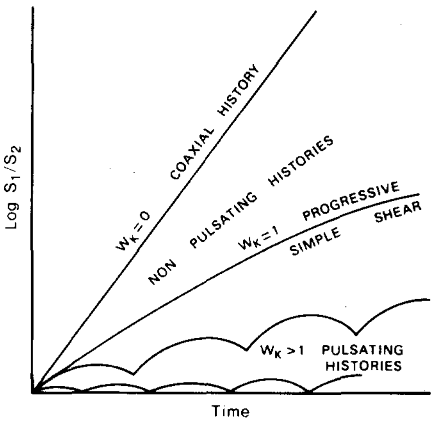

Fig. 3. Plot of logarithm of the ratio of the maximum to minimum principal stretches against time for some possible deformation histories. (cf. McKenzie 1979, fig. lb.) $W_k = 0$ corresponds to a coaxial history. All others are non-coaxial. $W_k = 1$ corresponds to progressive simple shear and the field between $W_k = 0$ and $W_k = 1$ comprises all non-pulsating histories. Pulsating deformation histories lie in the field below $W_k = 1$ and correspond to $1 < W_k < \infty$.

An example of a general three-dimensional deformation history is given by

$$x_1 = [\cos \alpha t X_1 + (aX_3 - bX_1) \sin \alpha t] \exp(-\beta t)$$
$$x_2 = X_2 \exp(\gamma t)$$
$$x_3 = [\cos \alpha t X_3 + (bX_3 - aX_1) \sin \alpha t] \exp(-\beta t).$$

This is inspired by equation (127) of Ramberg (1975). For isochoric histories $\gamma = 2\beta$ and $b^2 = a^2 - 1$. The velocity field for isochoric histories may be written in terms of the coordinates in the deformed state as

$$v_1 = [-(\alpha b + \beta)x_1 + a \alpha x_3]$$
$$v_2 = 2\beta x_2$$
$$v_3 = [-a \alpha x_1 + (\alpha b - \beta)x_3].$$

For such a history

$$W_k = \frac{-a\alpha}{(3\beta^2 + \alpha^2 b^2)^{1/2}}.$$

For coaxial histories, $W_k = 0$ whenever $a = 0$ or $\alpha = 0$
For non-pulsating histories, $0 < W_k < 1$ whenever $\alpha < \beta\sqrt{3}$
For progressive simple shearing, $W_k = 1$ whenever $\alpha = \beta\sqrt{3}$
For pulsating histories, $W_k > 1$ whenever $\alpha > \beta\sqrt{3}$
Ramberg (1975) chose in his equation (127) values of $\alpha = 0.446514$ and $\beta = 0.025$ so that $\alpha > \beta\sqrt{3}$ and a pulsating history develops.

DISCUSSION

The kinematical vorticity number should be a useful measure of non-coaxiality where one is dealing with what could be called steady-state fabrics or sub-fabrics. By this we mean fabrics that remain constant in character despite continuing deformation. Fabrics or fabric elements like this are expected in situations where low rates of strain or high temperature or other factors enable local fabric changes introduced by deformation to be balanced by opposite changes elsewhere, so that a given large volume of material has a constant population of each fabric element. In such situations the character of the fabric can be correlated best with instantaneous quantities, and the degree of non-coaxiality needs to be described in terms of history-independent measures like the kinematical vorticity number.

In other situations, where non-steady-state fabrics or subfabrics are developing, there is continuous change in or evolution of the fabric. Here different measures of non-coaxiality are more suitable. Two such measures suggested by Elliott (1972, p. 2628) are the angle between the stretching directions and the principal directions of finite strain so far accumulated and the angular velocity with respect to the λ_1 total finite strain direction of the material line instantaneously under the λ_1 direction. These are still instantaneous measures, as all measures of non-coaxiality ought to be, but they are now history-sensitive in the sense that they involve the total finite strain. They are also time-dependent. This time dependence is desirable where there is some corresponding time-dependence in the way that the fabric is responding to successive increments of deformation.

Finally, an attempt has been made in this paper to clarify the terminology attached to the general problem of rotations associated with deformations. In any deformation most lines rotate with respect to coordinate axes both internal and external to the body. Moreover material lines generally rotate at different rates to non-material lines. This leads to considerable confusion conceptually and the situation is not improved by the indiscriminate use of terms such as rotation, angular velocity, spin and vorticity to describe aspects of the displacement or velocity fields. This is the present situation in much of the modern continuum mechanics literature and in many instances where one is interested solely in a geometrical or kinematic description of the flow there is no need to make the kinds of distinctions made in this paper. However, if the interest lies in the development of fabric it is necessary to distinguish those aspects of the flow such as the internal vorticity that have a direct bearing on fabric development and evolution. Other aspects such as the spin component as defined here are irrelevant since they amount to a rigid rotation of the body as a whole. It is important to note that the kinematical vorticity number as defined and used here is meant to refer solely to those aspects of the flow that are relevant to fabric development.

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